More on Sphere Packing

So the ratio of filled space by sphere packing on a very regular looking lattice can be optimized to around 0.740 (as by using a Hexagonal Close-Packed lattice) and later with a second lattice of smaller spheres improved to 0.757. But optimization aside we can look at less regular filling methods. An interesting method is to place the spheres randomly and fill the space until no more spheres can be placed. These have an upper-bound of the same ratio of HCP and an average ratio of 0.64. Another interesting method is to do the same but also varying the radius of the spheres at random. While these two methods do not rely on a lattice the fill-ratio results can only be measured in averages and are especially uncertain in smaller spaces. The following method has a series—really a mathematical sequence—of lattices that fill a space in a very interesting way.

Another (irregular) Lattice for Sphere Packing)

The two basic principles behind what I will call the Prime Sequence Close-Packed lattice (P.S.C.P.) are (1) that certain numbers—primes— have
no non-trivial divisors and (2) that each sphere will be as big as it can be based on its location and the existing space available.

For convenience let’s set the goal to be filling a the volume of a 1x1x1 box with edges on the x-, y-, and z-axis. The location and size of each sphere will be designated by the cartesian coordinates of its center and its radius. This method does not have a clear end and can be continued to infinite hopefully reaching a maximum ratio (still to be determined). This ratio will probably be than HCP’s 0.740 because the spheres have the opportunity to vary in size and the sequence is endless. The sequence will converge to 1.0 at ∞.

Before beginning the placement of the spheres, the first step is to decide on a list of positive prime numbers. For simplicity, we’ll look at the sequence of prime numbers greater than one (2, 3, 5, 7, 11, 13, 17, 19, . . .). Since a list of primes could be theoretically endless and likewise the number of steps in this method, the actions on each “level” will be what’s most important.

P.S.C.P. — Level One

Taking level zero to be our empty box with fill ratio 0.0, level one will assign variable $p = 2$, our first prime number in the list. Form a lattice in the box with intersections at every $1/p$ units. In the case of level one we make a lattice with intersections at every $1/2$ units. Doing this we can that there are $(2 + 1)^3 = 27$ intersections—some, actually most, lie on the edge, but that will be addressed. At each intersection we will try to blow a balloon so to speak, taking the intersection to be the center of our balloon sphere. Our balloon will stop growing in radius if its surface touches the boundaries of the box or another existing balloon. So 26 of the 27 insections on this level will have radius zero (they popped, well, they never existed in the first
place...can a balloon with zero volume really pop?), leaving only the one intersection in the middle. At this intersection on the lattice we can blow a balloon with radius \( r = \frac{1}{p} = \frac{1}{2} \). Since we exhausted every lattice point on level one we can move on to level two.

P.S.C.P. — Level Two

Now we will assign variable \( p = 3 \), the next prime number in the list. Form a new lattice now with intersections at every \( 1/p = 1/3 \) units. We see that there will be \( 4^3 = 64 \) intersections total, but like before only some do not lie
on the edges of the box. There \((p-1)^3 = 2^3 = 8\) completely inside the space. However, no balloons can be blown at any of these points either because all lie within an existing balloon and therefore all would have negative radius (a negative volume balloon? Could it also pop?). On to level three.

\[3\]

**P.S.C.P. — Level Three**

Again we will assign variable \(p = 5\), the next prime number in the list. Form again a new lattice now with intersections at every \(1/p = 1/5\) units giving \(6^3 = 216\) intersections total, but \(4^3 = 64\) of interest. Some of which will
not grow balloons because they lie in the existing sphere, but finally a few more that lie in the free space. Those (eight) lattice points outside existing spheres our boundaries grow to final radii, maximized until reaching the surface of the existing balloon.

P.S.C.P. — Level Four

We repeat the process and see that now with $p = 7$ we have new spheres on the new lattice. The spheres from level four have varying radii depending on the conditions of their respective locations. The space is beginning to
P.S.C.P. — Level $n$

At any given level we take $p = p_n$ the $n^{th}$ prime in our list. We get a lattice grid with $(n + 1)^3$ intersections total, but with only $(n - 1)^3$ inside the box properly. Of the remaining, depending on the conditions some spheres will grow and grow maximally.
Why primes?

If each level is based on a single prime number we are guaranteed that the intersections created will not repeat any previously attempted intersections from previously used lattices. For counter example look at the repeat points when using 2 and then 4. Choosing only primes is then an efficiency choice and not necessarily a principle choice since 4 does provide some points not reached by any primes. The packing achieved by P.S.C.P. will fill without those points.
Conclusion

The limit should be 1.00 but at infinite.

Appendix

A few more photos and variations on this method.

The method run over a sequence of 10 integers.
The method run over a sequence of 17 integers.
The nine primes greater than two.