Notes on Inflating Curves

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Generating an inflated shape from a sketched curve [2, 7, 6, 3] leads to the following problem:

Problem Given a planar region $\Omega \subset \mathbb{R}^2$, specified by its boundary curve, generate a height field $z(x, y)$ defined on Ω, such that z is zero on $\partial\Omega$ and the graph of $\pm z$ forms a "nice" smooth surface.

To get the obvious out of the way, it is clear that away from the boundary, z must be smooth. On the boundary, smoothness requires that the surface normal be in the xy plane (otherwise, there is a crease).

Solution We have observed that solving the Poisson equation $\nabla^2 h(x, y) = -4$, subject to $h(\partial\Omega) = 0$ and setting $z = \sqrt{h}$ produces very nice results (Figure 1) and is relatively fast and easy to compute. We are curious as to why the results look good.

Our method clearly satisfies the conditions above: smoothness of z is obvious. The fact that the derivative of the square root function approaches infinity as its argument approaches zero ensures that as long as the gradient of h does not vanish on $\partial\Omega$, the tangent plane to the surface $(x, y, z(x, y))$ on $\partial\Omega$ is orthogonal to the xy plane.

Special Cases Let's consider two special cases that we can solve analytically. If Ω is a unit disk, the function $h = 1 - x^2 - y^2$ satisfies the Poisson equation and the boundary conditions and the inflated surface $\pm\sqrt{1 - x^2 - y^2}$ is a sphere. If Ω is the region $-1 \leq y \leq 1$, the function $h = 2 - 2y^2$ is the solution and the inflated surface is a cylinder whose cross-section is an ellipse with major radius $\sqrt{2}$ and minor radius 1.

RMS Distance The method originally came about from the following consideration: the puffier parts of a pillow are farther away from the boundary. Taking z to be a distance field from $\partial\Omega$ would not work because it would not be smooth at the medial axis (and at the boundary because the gradient of z would be finite). So to find $z(x, y)$, instead of the minimum distance to $\partial\Omega$, let's take the mean distance, but weigh it by generalized barycentric coordinates of (x, y) with respect to $\partial\Omega$.

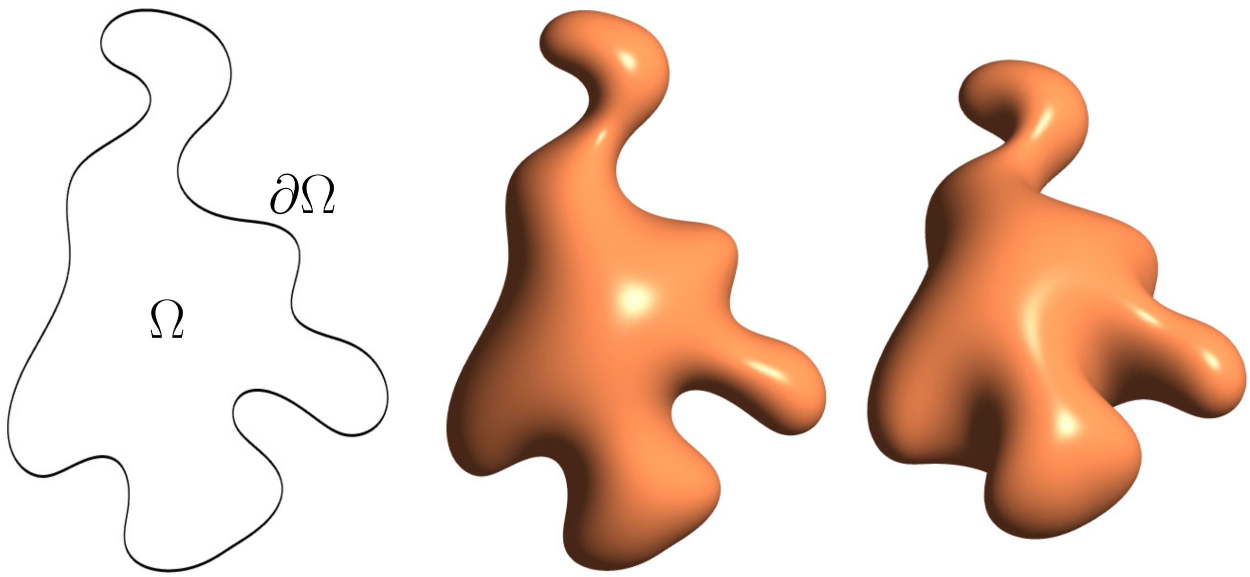

Figure 1: The curve (left) is inflated into a surface (middle). On the right is a rotated view of the result.

Given a function $f : \partial\Omega \to \mathbb{R}$, define $f^C : \Omega \to \mathbb{R}$ to be the extension of f to Ω's interior using generalized barycentric coordinates C (so use f^H for harmonic coordinates [4] and f^M for mean value coordinates [1]). In particular, at a point $p \in \Omega$, whose coordinate with respect to a point $b \in \partial\Omega$ is $C_p(b)$, $f^C(p) = \int_{\partial\Omega} f(b) C_p(b)\, db$.

Let $d_{x_0,y_0} : \partial\Omega \to \mathbb{R}$ be the distance from points on the boundary to (x_0, y_0). A possible scheme is then $z(x, y) = d^C_{x,y}(x, y)$. This is nice and smooth in the interior, but has a crease along the boudary. Additionally, it is somewhat expensive to compute, because a different interpolant needs to be computed for every point. A square root would fix the smoothness problem, and if z is to have distance units, we get the modification: $z(x, y) = \sqrt{\left(d^2_{x,y}\right)^C(x, y)}$. Expand $d^2_{x_0,y_0} = (x - x_0)^2 + (y - y_0)^2 = x^2 + y^2 - 2xx_0 - 2yy_0 + x_0^2 + y_0^2$. Because generalized barycentric interpolation is linear and preserves linear and constant functions, we get: $\left(d^2_{x_0,y_0}\right)^C(x, y) = \left(d^2_{0,0}\right)^C(x, y) - 2xx_0 - 2yy_0 + x_0^2 + y_0^2$. Therefore, $z = \sqrt{\left(d^2_{0,0}\right)^C(x, y) - x^2 - y^2}$. If we choose harmonic coordinates, to compute z, we only need to solve the Laplace equation for $\left(d^2_{0,0}\right)^H$. It has the form $\nabla^2(h+x^2+y^2) = 0$ with the boundary condition $(h+x^2+y^2) = x^2+y^2$ on $\partial\Omega$. This is equivalent to our original equation. Mean value coordinates can also be used, but the computation is slower (computing the interpolation is no longer a single linear solve) and the surfaces do not look as nice.

Known Energy Functionals The standard method for generating fair surfaces is by minimizing an energy functional. For example, if κ_1 and κ_2 are the principal curvatures on a surface S, the surface's Willmore energy is defined to be $\int_S (\kappa_1 - \kappa_2)^2\, dA$. The variation of curvature (MVC) energy [5] is $\int_S \left(\frac{d\kappa_1}{de_1}\right)^2 + \left(\frac{d\kappa_2}{de_2}\right)^2 dA$, where e_1 and e_2 are the principal curvature directions. The case when Ω is a disk does not rule out either of these energy functionals: both are zero for a sphere. However, the second special case when Ω is a strip rules out both: the cross-section of the cylinder that minimizes MVC energy is a circle, while the cross-section of the cylinder that minimizes Willmore energy is a minimum bending energy curve and is not an ellipse. Our elliptical cross section rules out many variations of the above functionals: translational symmetry implies that the cross section is all that can be optimized and there are not many different curve energy functionals.

Our Energy Functional We have found that our surface minimizes

$$\int_\Omega \|\nabla(x^2 + y^2 + z^2)\|^2 dA$$

over possible values of z (the gradient is taken with respect to x and y). Let $h = z^2$ and let h_0 be a perturbation, with $h_0(\partial\Omega) = 0$. The variational derivation is:

$$\begin{aligned}
\text{(substitute } h\text{)} \qquad 0 &= \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \int_\Omega \|\nabla(x^2 + y^2 + (h + \epsilon h_0))\|^2 dA \\
\text{(expand product and remove zero terms)} \qquad &= \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \int_\Omega 2\epsilon \nabla h_0 \cdot \nabla(x^2 + y^2 + h) dA \\
\text{(differentiate and divide by 2)} \qquad &= \int_\Omega \nabla h_0 \cdot \nabla(x^2 + y^2 + h) dA \\
\text{(apply divergence product rule)} \qquad &= \int_\Omega \nabla \cdot (h_0 \nabla(x^2 + y^2 + h)) dA - \\
&\quad - \int_\Omega h_0 \nabla^2(x^2 + y^2 + h) dA \\
\text{(apply divergence theorem and simplify)} \qquad &= -\int_\Omega h_0(4 + \nabla^2 h) dA
\end{aligned}$$

The fundamental lemma of variational calculus therefore implies that $4 + \nabla^2 h = 0$, which is the equation we solve.

Interestingly, the energy functional is not invariant to the translation of Ω, although the shape of its minimum clearly is (being the solution of a translation-invariant equation). The functional is also strongly tied to the surface parameterization: if we let $S(u, v) = (x(u, v), y(u, v), z(u, v))$ with $(u, v) \in \Omega$ and optimize $\int_\Omega \|\nabla(S \cdot S)\|^2 dA$ subject to the constraint that on $\partial\Omega$, $S = (u, v, 0)$, there is a continuum of critical points, including a flat one ($z = 0$). Perhaps an additional condition can be found that generalizes the optimization to parametric surfaces, not just height fields.

References

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